ρ 的性质，在 Bloch 球里的含义

引　$\boxed{A}\leftrightarrow B$　open system．有噪声的系统．

放一个 $|\psi\rangle_A$ 进去　出来一个 $|\psi\rangle_{A'}$ 混态，B 对 A 产生影响，出来 A，不是纯态，用 ρ' 来描述

叫做 quantum channel．

下面就来一起得到这个 q.c. 的详细形式．

Overview　如何研究呢？环境 B 是对进行了测量（不知道多少次），但测量结果不得而知。

从简单的情况讲起，即 $\boxed{S \leftrightarrow A}$（system　apparatus）　系统 S 与仪器 A 耦合 然后测量 A．

暂时只 S，A．而不 Alice Bob．

分析测量结果：$|\psi\rangle_{SA} = \sum_a \sqrt{p_a}\,|\phi_a\rangle_S \otimes |\mu_a\rangle \xrightarrow{\text{测量}}$ p_a 的概率 $|\phi_a\rangle_S$，ρ_a'（纯态）　$p_a \rightarrow \rho_a'$　$|\phi_a\rangle\langle\phi_a|$　$\sum_a p_a\rho_a$

不知道这一次 A 的测量结果，只知道进行了一次测量，那么最终 S 变成如何？

—— 于是一种经典的概率相加　p_a 概率处于 ρ_a'　于是 S 被描述为　$\rho' = \sum_a p_a \rho_a'$　（可能是混态）

现在问题只剩下，ρ_a' 为何？

Quantum Measurement

$\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$

S 中的一组正交算符　$\{E_a\}$：

$$\begin{cases} E_aE_b = \delta_{ab}E_a & \text{正交} \\ E_a = E_a^\dagger & \text{厄米} \\ \sum_a E_a = I & \text{和为“1”} \end{cases}$$

对于 A 系统（d dimensional）　在正交归一基 $\{|a\rangle\}$　$(a = 0,1,2\cdots d-1)$

引入一个作用在 SA 系统的幺正算符：$U_{SA} = \sum_a\sum_b E_a \otimes |b+a\rangle\langle b| = \sum_{a,b} E_a \otimes |b+a\rangle\langle b|$

U_{SA} 的作用效果，对 apparatus 初始化到 $|0\rangle$，即 SA 态：$|\Phi\rangle_{SA} = |\psi\rangle \otimes |0\rangle$

$$U_{SA}|\Phi\rangle_{SA} = \Big(\sum_{a,b} E_a \otimes |b+a\rangle\langle b|\Big)\big(|\psi\rangle \otimes |0\rangle\big) = \sum_{a,b} E_a|\psi\rangle \otimes \underbrace{|b+a\rangle\langle b|0\rangle}_{|0+a\rangle} = \sum_a E_a|\psi\rangle \otimes |a\rangle$$

即．$U: |\psi\rangle\otimes|0\rangle \rightarrow \sum_a E_a|\psi\rangle \otimes |a\rangle \Rightarrow \mathrm{Prob}(a) = \langle\, I \otimes |a\rangle\langle a|\,\rangle = \langle\psi|E_a|\psi\rangle = \|E_a|\psi\rangle\|^2$

$\sum_a \sqrt{p}\,|s\rangle \otimes |a\rangle$　“Schimit 分解”

验证幺正性：
$$\begin{aligned} UU^\dagger &= \Big(\sum_{a,b} E_a \otimes |b+a\rangle\langle b|\Big)\Big(\sum_{c,d} E_c \otimes |d\rangle\langle d+c|\Big) \\ &= \sum_{a,b,c,d} (E_aE_c) \otimes \big(|b+a\rangle\langle b|d\rangle\langle d+c|\big) \\ &= \sum_{a,b,c,d} \delta_{ac}\,\delta_{bd}\, E_a \otimes |b+a\rangle\langle d+c| \\ &= \sum_a\sum_b E_a \otimes |b+a\rangle\langle b+a| \\ &= \sum_a E_a \otimes I = I \otimes I \end{aligned}$$

Generalized Measurements

回到 Alice, Bob.

前面 $U_{SA}: |\psi\rangle\otimes|0\rangle \to \sum_a E_a|\psi\rangle\otimes|a\rangle$

不一定 $E_a|\psi\rangle$ 一般的对 $M_\mu|\psi\rangle$ $\quad U = \sum_{\mu, b} M_\mu \otimes |b+\mu\rangle\langle b|$

$U: |\psi\rangle_A \otimes |0\rangle_B \to |\psi'\rangle_{AB} = \sum_\mu M_\mu|\psi\rangle_A \otimes |\mu\rangle_B$

必须有 $UU^\dagger = 1 \Rightarrow 1 = \sum_{\mu,\nu} (\langle\psi|M_\mu^\dagger \otimes \langle\mu|)(M_\nu|\psi\rangle\otimes|\nu\rangle) = \sum_\mu \langle\psi|M_\mu^\dagger M_\mu|\psi\rangle \Rightarrow \boxed{\sum_\mu M_\mu^\dagger M_\mu = I_A}$

对 B 的测量 $\{E_\mu = I_A \otimes |\mu\rangle\langle\mu|\}$

$p_\mu = \langle E_\mu\rangle = \langle\psi'|E_\mu|\psi'\rangle = {}_A\langle\psi|M_\mu^\dagger M_\mu|\psi\rangle_A$

测量后的态为 $\dfrac{E_\mu|\psi'\rangle_{AB}}{\|E_\mu|\psi'\rangle_{AB}\|} = \dfrac{M_\mu|\psi\rangle_A}{\|M_\mu|\psi\rangle_A\|} \otimes |\mu\rangle$

若 M_μ 正交: $\quad M_\nu M_\mu = \delta_{\nu\mu} M_\mu \times \text{Phase}. \Rightarrow P(\nu|\mu) = \dfrac{\|M_\nu M_\mu|\psi\rangle_A\|^2}{\|M_\mu|\psi\rangle_A\|^2} = \delta_{\mu\nu}$

即无论之后如何测量 结果不变, 状态不变

总结: Positive Operator-Valued Measure (POVM) [Generalized Measurements]

对一个 $|\psi\rangle \longrightarrow |\psi\rangle\otimes|0\rangle \xrightarrow{\text{幺正变换}} |\Phi\rangle_{AB} \xrightarrow{\text{B正交归一基测量}} \dfrac{M_\mu|\psi\rangle_A}{\|M_\mu|\psi\rangle_A\|}$

$E_a = M_a^\dagger M_a$

$$\begin{cases} E_a = E_a^\dagger \\ \langle\psi|E_a|\psi\rangle = \langle\psi|M_a^\dagger M_a|\psi\rangle \geq 0 \\ \sum_a E_a = \sum_a M_a^\dagger M_a = I \end{cases}$$

于是 $p_a = \langle\psi|E_a|\psi\rangle = \mathrm{tr}\, \rho E_a$

普遍有任意 $M_a = U_a\sqrt{E_a}$

测后态 $U_a \dfrac{\sqrt{E_a}|\psi\rangle}{\|\sqrt{E_a}|\psi\rangle\|}$

$$|\phi\rangle \to M_\mu|\phi\rangle$$

$$\rho_0 = |\phi\rangle\langle\phi| \to M_\mu|\phi\rangle\langle\phi|M_\mu^\dagger / \mathrm{tr}(\sim)$$

$$\rho = \sum_b \lambda_b |\phi_b\rangle\langle\phi_b| \to \sum_b \lambda_b M_\mu|\phi_b\rangle\langle\phi_b|M_\mu^\dagger/\mathrm{tr}(\sim) = M_\mu \rho M_\mu^\dagger / \mathrm{tr}(\sim)$$

即 $\rho_a' = \dfrac{M_a \rho M_a^\dagger}{\mathrm{tr}\, M_a \rho M_a^\dagger}$

$$\rho' = \sum_a p_a \rho_a' = \sum_a \mathrm{tr}\, M_a \rho M_a^\dagger \frac{M_a \rho M_a^\dagger}{\mathrm{tr}\, M_a \rho M_a^\dagger} = \sum_a M_a \rho M_a^\dagger, \quad \sum_a M_a^\dagger M_a = I$$

抽象 $\rho' = \mathcal{E}(\rho) = \underbrace{\sum_a M_a \rho M_a^\dagger}_{\text{Kraus representation 不唯}}$ quantum channel / trace-preserving completely positive map (TPCP map)