1 Phase shifts for hard sphere scattering

(a). Find the phase shifts for scattering by a hard sphere

$$V(r) = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases} \tag{1}$$

(b) Find the total cross section for an incoming energy

$$E = \hbar^2 k^2 / 2m \tag{2}$$

in the two limits

k→∞, k→0

Give a physical interpratation of the factor 4 and 2 in your answers.

Hint 1:For $k \to \infty$ use the asymptotic form of j_l and n_l to obtain a simple form for sinδ_l^2

Hint 2 : Look at Zettili,Problem 11.3

Solution (a)

In this problem we need not even evaluateβ_l(which is actually ∞). All we need to know is that the wave function must vanish at r=R because the sphere is impenetrable.

Therefore:

$$A_l(r)|_{r=R} = 0$$

or, from $A_l(r)|_{r=R} = e^{i\delta_l}[\cos\delta_l j_l(kR) - \sin\delta_l n_l(kR)]$,

$$\cos\delta_l j_l(kR) - \sin\delta_l n_l(kR) = 0$$

or,

$$\tan\delta_l = \frac{j_l(kR)}{n_l(kR)}$$

Thus the phase shifts are known for any l.

Notice that no approximations have been made so far.

(b) k→0, kr≪1

use $j_l(kr) \approx \frac{(kr)^l}{(2l+1)!!}, (kr \to 0)$

$n_l(kr) \approx -\frac{(2l-1)!!}{(kr)^{l+1}}, (kr \to 0)$

to obtain

$$tan\delta_l = \frac{-(kr)^{2l+1}}{(2l+1)[(2l-1)!!]^2}$$

It is therefore all right to ignore δ_l with l≠0

In other words,we have s-wave scattering only,which is actually expected for almost any finite-range potential at low energy.

$\tan\delta_l = \frac{j_l(kR)}{n_l(kR)}$

for $l = 0$, $\tan\delta_0 = \frac{j_0(kR)}{n_0(kR)} = \frac{\sin(kR)/kR}{-\cos(kR)/kR} = -\tan(kR)$

$\delta_0 = -kR$

$f(\theta) = \frac{1}{k}\sum_{l=0}(2l+1)e^{i\delta_l}\sin\delta_l p_l(cos\theta)$

$\frac{d\sigma}{d\omega} = |f(\theta)|^2$

for $l = 0$, $\frac{d\sigma}{d\Omega} = \frac{\sin^2\delta_0}{k^2}$

for $kr \ll 1$, $\frac{d\sigma}{d\Omega} = \frac{\sin^2(k^2R^2)}{k^2} \approx \frac{k^2R^2}{k^2} = R^2$

the total cross section ,given by

$$\sigma_{tot} = \int \frac{d\sigma}{d\Omega} d\Omega = 4\pi R^2$$

is four times the geometric cross section πR^2.

这可以理解为衍射效应：在经典力学中，在低能极限下（$k \to 0$, 因而 $\lambda \to \infty$），障碍物的尺度远小于波长 λ, 入射波发生衍射。因为 s 波是各向同性的（$l = 0$,$m = 0$,Y_{00} 与 θ,ϕ 无关），因此硬球表面各处对散射有同等贡献，总散射截面等于刚球的表面积 $4\pi r^2$.

$k \to \infty$, $ka \gg 1$

the number of partial waves contributing to the scattering is large.We may regard l as a continuous variable.

note:

$\overrightarrow{L} = \overrightarrow{r} \times \overrightarrow{p}$,

$\sqrt{l(l+1)}\hbar \sim rp = rk\hbar$,

$\sqrt{l(l+1)} \sim rk$,

$r \sim b$, $k \to \infty$, $l \to \infty$

as an aside,we note the semiclassical argument that l=bk

note :

$\sqrt{l(l+1)}\hbar \sim bp$, for l large, $l\hbar \sim bp$

两边除以 $\hbar$, 就得到 l=bk

we take $l_{max} = kR$,

at high energies many l-values contribute,up to $l_{max} = kR$,a reasonable assumption. The total cross section is therefore given by

$\sigma_{tot} = \frac{4\pi}{k^2} \sum\limits_{l=0}^{l \approx kR} (2l+1) sin^2 \delta_l$

note:

$f(\theta) = \frac{1}{k} \sum\limits_{l=0} (2l+1) e^{i\delta_l} \sin\delta_l p_l(cos\theta)$

$\sigma_{tot} = \int |f(\theta)|^2 d\Omega$

$= \frac{1}{k^2} \int_0^{2\pi} d\phi \int_{-1}^{+1} d(cos\theta) \sum\limits_{l} \sum\limits_{l'} (2l+1)(2l'+1) \times e^{i\delta_l} \sin\delta_l e^{-i\delta_{l'}} \sin\delta_{l'} P_l P_{l'}$

$= \frac{4\pi}{k^2} \sum\limits_{l} (2l+1) sin^2 \delta_l$

勒让德多项式的正交归一关系式

$\int_{-1}^{+1} P_l(x) P_k(x) dx = \frac{2}{2l+1} \delta_{lk}$

$\sin^2\delta_l = \tan^2\delta_l \cos^2\delta_l = \frac{\tan^2\delta_l}{\sec^2\delta_l} = \frac{tan^2\delta_l}{1+tan^2\delta_l} = \frac{[j_l(kR)]^2}{[j_l(kR)]^2+[n_l(kR)]^2}$,

use the asymptotic behavior when kr→∞,

$j_l(kr) \sim \frac{1}{kr} \sin(kr - \frac{l\pi}{2})$

$n_l(kr) \sim -\frac{1}{kr} \cos(kr - \frac{l\pi}{2})$

we obtain

$\sin^2\delta_l \sim sin^2(kR - \frac{\pi l}{2})$

each time l increases by one unit, δ_l decreases by $\frac{\pi}{2}$.

Thus,for an adjacent (相邻的) pair of partial waves,

$\sin^2\delta_l + sin^2\delta_{l+1} = sin^2\delta_l + sin^2(\delta_l - \frac{\pi}{2}) = sin^2\delta_l + \cos^2\delta_l = 1$

and with so many l-values contributing to

$\sigma_{tot} = \frac{4\pi}{k^2} \sum_{l=0}^{l\approx kR} (2l+1) \sin^2\delta_l$,

it is legitimate to replace $\sin^2\delta_l$ by its average value, $\frac{1}{2}$

The number of terms in the l-sum is roughly kR,

and the average of $2l+1$ is roughly kR,

$\sigma_{tot} \approx \frac{4\pi}{k^2}(kR)(\frac{1}{2}kR) = 2\pi R^2$

note:

如果用求和公式来算,

$\frac{1}{2}\frac{4\pi}{k^2} \sum\limits_{l=0}^{l\approx kR} (2l+1) = \frac{2\pi}{k^2}[1+(2kR+1)](kR+1)\frac{1}{2} = \frac{2\pi}{k^2}(kR+1)^2$,

$(kR \gg 1)$, 上式 $\approx 2\pi R^2$

To see the origin of the factor of 2,we may split

$f(\theta) = \sum\limits_{l=0} (2l+1)(\frac{e^{2i\delta_l}-1}{2ik}) P_l(\cos\theta) = \frac{1}{k} \sum\limits_{l=0} (2l+1) e^{i\delta_l} \sin\delta_l P_l(cos\theta)$

into two parts:

$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{kR}(2l+1)e^{2i\delta_l}P_l(\cos\theta) + \frac{i}{2k}\sum_{l=0}^{kR}(2l+1)P_l(\cos\theta)$
$=f_{reflection} + f_{shadow}$

In evaluating $\int |f_{refl}|^2 d\Omega$,the orthogonality of the $P_l(\cos\theta)'s$ ensure that there is no interference among contributions from different l,and we obtain the sum of the square of partial-wave contributions:

$\int |f_{refl}|^2 d\Omega = \frac{2\pi}{4k^2}\sum_{l=0}^{l_{max}}\int_{-1}^{+1}(2l+1)^2[P_l(\cos\theta)]^2 d(\cos\theta) \approx \frac{\pi l_{max}^2}{k^2} = \pi R^2$

note: $\int_{-1}^{+1} P_l(x)P_k(x)dx = \frac{2}{2l+1}\delta_{lk}$

Turning our attention to f_{shad},it is pure imaginary. It is particularly strong in the forward direction because $P_l(\cos\theta) = 1$ for $\theta = 0$, $(P_l(1) = 1)$

and the contributions from various l-values all add up coherently ,that is,with the same phase, pure imaginary and positive in our case.

We can use the small-angle approximation for P_l to obtain

$f_{shad} \approx \frac{i}{2k}\sum(2l+1)J_0(l\theta)$
$\approx \frac{i}{2k}\int(2bk+1)J_0(kb\theta)kdb$
$\approx ik\int_o^R bdbJ_0(kb\theta)$
$= \frac{iRJ_1(kR\theta)}{\theta}$

this is just the formula for Fraunhofer diffraction in optics with a strong peaking near θ=0.

Letting $\xi = kR\theta$ and $d\xi/\xi = d\theta/\theta$,
we can evaluate $\int |f_{shad}|^2 d\Omega = 2\pi\int_{-1}^{+1}\frac{R^2[J_1(kR\theta)]^2}{\theta^2}d(\cos\theta)$
$\approx 2\pi R^2\int_\pi^0 \frac{[J_1(\xi)]^2}{\theta^2}(-\sin\theta)d\theta$
$\approx -2\pi R^2\int_\infty^0 \frac{[J_1(\xi)]^2}{\xi}d\xi$
$\approx 2\pi R^2\int_0^\infty \frac{[J_1(\xi)]^2}{\xi}d\xi$
$\approx \pi R^2$

Finally, the interaction between f_{shad} and f_{refl} vanishes,
because the phase of f_{refl} oscillates $(2\delta_{l+1} = 2\delta_l - \pi)$,
approximately averaging to zero,while f_{shad} is pure imaginary.

$f(\theta) = \frac{1}{2ik}\sum_{l=0}^{kR}(2l+1)e^{2i\delta_l}P_l(\cos\theta) + \frac{i}{2k}\sum_{l=0}^{kR}(2l+1)P_l(\cos\theta)$
$=f_{reflection} + f_{shadow}$
$e^{2i\delta_l} = \cos 2\delta_l + i\sin 2\delta_l$,
$2\delta_{l+1} = 2\delta_l - \pi$
note:

pure imaginary number × oscillates $\rightarrow$ oscillates ,

$\sum_l oscillates \quad \rightarrow 0$

End note

$Re(f^*_{shad} f_{refl}) \approx 0$

$\sigma_{tot} = \int (f^*_{refl} + f^*_{shad})(f_{refl} + f_{shad})\ \mathrm{d}\Omega$

$= \int f^*_{shad} f_{shad} d\Omega + \int f^*_{shad} f_{refl} d\Omega + \int f^*_{refl} f_{shad} d\Omega + \int f^*_{refl} f_{refl} d\Omega$

$= \int |f_{shad}|^2 d\Omega + \int |f_{refl}|^2 d\Omega + \int 2Re(f^*_{shad} f_{refl}) d\Omega$

the interference between f_{shad} and f_{refl} vanishes.

Thus, $\sigma_{tot} = \pi R^2(\sigma_{refl}) + \pi R^2(\sigma_{shad})$

第二项（朝前方的相干贡献）叫做阴影，因为对于高能硬球散射，碰撞参数小于 R 的波一定会被偏转，所以，就在靶的后方找到粒子的概率是 0，一定会产生一个阴影。

从波动力学来说，这个阴影是由于最初的入射波和新散射的波之间的非常强的干涉（destructive interference）, 因此我们需要散射来产生一个阴影。

阴影散射振幅 f_{shad} 一定是纯虚数可以从下面看出：

$< \overrightarrow{x} | \psi^{(+)} > \quad \rightarrow (large \quad r)$

$\frac{1}{(2\pi)^{3/2}} [e^{ikz} + f(\theta) \frac{e^{ikr}}{r}]$

$= \frac{1}{(2\pi)^{3/2}} \sum_l (2l+1) \frac{p_l}{2ik} [[1 + 2ikf_l(k)] \frac{e^{ikr}}{r} - \frac{e^{-i(kr - l\pi)}}{r}]$

that the coefficient of $e^{ikr}/2ikr$ for the lth partial wave behaves like $1 + 2ikf_l(k)$,

where the l would be present even without the scatterer,

hence there must be a positive imaginary term in f_l to get cancellation.

note:

$f_l --- f_{shad}$ 的关系

$f(\theta) = \sum_{l=0} (2l+1) (\frac{e^{2i\delta_l} - 1}{2ik}) P_l(\cos\theta)$,

$f_l(\theta) = \frac{e^{2i\delta_l} - 1}{2ik} = \quad \frac{e^{2i\delta_l}}{2ik}$ （振荡）$+ \frac{i}{2k}$ （正虚数），

$[1 + 2ikf_l(k)] < 1, \rightarrow f_l$ 近似于一个正虚数

In fact ,this gives a physical interpretation of the optical theorem, which can be checked explicitly.

First note that

$\sigma_{tot} = \frac{4\pi}{k} I_m f(\theta = 0)$

因为 f_{refl} 振荡，$I_m[f_{refl}(0)]$ averages to zero due to oscillating phase.

所以

$$\frac{4\pi}{k} I_m f(\theta = 0) \approx \frac{4\pi}{k} I_m f_{shad}(\theta = 0)$$

,

Using

$f(\theta) = \frac{1}{2ik} \sum_{l=0}^{kR} (2l+1) e^{2i\delta_l} P_l(\cos\theta) + \frac{i}{2k} \sum_{l=0}^{kR} (2l+1) P_l(\cos\theta)$

$= f_{reflection} + f_{shadow}$,

we obtain

$$\frac{4\pi}{k} I_m f(\theta = 0) = \frac{4\pi}{k} \frac{1}{2k} \sum_{l=0}^{kR} (2l+1) = \frac{2\pi}{k^2} (kR+1)^2 \approx 2\pi R^2, \qquad (kR \gg 1)$$