Basics of Direct Nuclear Reaction

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1 Big picture of a direct reaction

1.1 Reaction

The reaction takes the form:

$$a \ + \ A \rightarrow c \ + C \tag{1}$$

where a is called the projectile and A is called the target. The character c stands for 'channel'. Channel is a key concept in direct nuclear reaction. The interaction between two nuclei can produce different partitions of the a+A system. Besides, the internal state may be excited. These are all different channels of the reaction.

1.2 Notations

The total Hamiltonian of the system is written as:

$$H = H_A + H_a + T_\alpha + V_\alpha \tag{2}$$

where α stands for the incident channel.

The internal states are represented by:

$$H_A \Phi_A = \epsilon_A \Phi_A, \quad H_a \Phi_a = \epsilon_a \Phi_a \tag{3}$$

But the Hamiltonian of the whole system can be expressed in different channels:

$$H = H_B + H_b + T_\beta + V_\beta \tag{4}$$

T and V represents the relative motion and potential between two nuclei.

In general, the potential is a sum over all nucleons:

$$V_\alpha = \sum_{i \in a, j \in A} V_{ij} \quad \text{or} \quad V_\beta = \sum_{i \in b, j \in B} V_{ij} \tag{5}$$

For convenience, we denote the total intrinsic Hamiltonian of α channel as:

$$H_\alpha = H_A + H_a \tag{6}$$

So the total intrinsic state can be described by:

$$H_\alpha \Phi_\alpha = \epsilon_\alpha \Phi_\alpha \tag{7}$$

2 Integral equation

The Schrodinger equation for the whole system is:

$$(E - H)\Psi = 0 \tag{8}$$

In a particular channel, this equation reads:

$$(E - H_\alpha - T_\alpha)\Psi_\alpha = V_\alpha \Psi_\alpha \tag{9}$$

We then consider the free solution:

$$(E - H_\alpha - T_\alpha)\phi_\alpha = 0 \tag{10}$$

this equation can be solved through the separation of variables, so the solution takes the form:

$$\phi_\alpha = \exp\left(i\boldsymbol{k_\alpha} \cdot \boldsymbol{r_\alpha}\right) \times \Phi_\alpha \tag{11}$$

where

$$\frac{\hbar^2 k_\alpha^2}{2\mu} = E_\alpha = E - \epsilon_\alpha \tag{12}$$

is the kinetic energy of the relative motion. We introduce two different inner product notations:

$$(\Phi_\alpha, \Psi_\alpha) = \int \Phi_\alpha^* \Psi_\alpha \mathrm{d}(\text{internal coordinates}) \tag{13}$$

$$\langle \phi_\alpha | \Psi_\alpha \rangle = \int \phi_\alpha^* \Psi_\alpha \mathrm{d}(\text{all coordinates}) \tag{14}$$

Since Ψ_α represents the state of the whole system, the relative motion state can be expressed by:

$$\psi_\alpha = (\Phi_\alpha, \Psi_\alpha) \tag{15}$$

this is because that all the internal coordinates are integrated out. Project the total state onto intrinsic state of α channel, the Schrodinger equation then reads:

$$(E_\alpha - T_\alpha)(\Phi_\alpha, \Psi_\alpha) = (\Phi_\alpha, V_\alpha \Psi_\alpha) \tag{16}$$

The equation of other channels can be obtained straightforward:

$$(E_\beta - T_\beta)(\Phi_\beta, \Psi_\alpha) = (\Phi_\beta, V_\beta \Psi_\alpha) \tag{17}$$

The reason why Ψ_α is still denoted with α is that the incident channel is represented by α.

2.1 T-matrix

Our goal is to solve the Schrodinger equation of the relative motion:

$$(E - H_\alpha - T_\alpha)\psi_\alpha = (\Phi_\alpha, V_\alpha \Psi_\alpha) \tag{18}$$

So we define the Green's function:

$$(E_\alpha - T_\alpha) G_\alpha^0(\boldsymbol{r_\alpha} - \boldsymbol{r'_\alpha}) = \delta(\boldsymbol{r_\alpha} - \boldsymbol{r_{\alpha'}}) \tag{19}$$

the asymptotic solution of Green's function is:

$$G_\alpha^{0(+)}(\boldsymbol{r},\ \boldsymbol{r'}) = -\frac{2\mu}{4\pi\hbar^2} \frac{e^{ik_\alpha r}}{r} \exp\left(-i\boldsymbol{k'_\alpha} \cdot \boldsymbol{r'}\right) \tag{20}$$

the plus sign represents the little positive imaginary part of in the denominator, which stands for the outgoing spherical solution.

The scattering amplitude can be expressed as:

$$f_{\alpha\alpha}(\theta) = -\frac{2\mu_\alpha}{4\pi\hbar^2}\int \mathrm{d}^3r' \exp\left(-i\boldsymbol{k'_\alpha}\cdot\boldsymbol{r'}\right)(\Phi_\alpha, V_\alpha\Psi_\alpha^{(+)}) \tag{21}$$

similarly, the scattering amplitude for other channels is:

$$f_{\beta\alpha}(\theta) = -\frac{2\mu_\beta}{4\pi\hbar^2}\int \mathrm{d}^3r' \exp\left(-i\boldsymbol{k'_\beta}\cdot\boldsymbol{r'}\right)(\Phi_\beta, V_\beta\Psi_\alpha^{(+)}) \tag{22}$$

Finally we can express the scattering amplitude in a more general and compact way, by defining the T-matrix:

$$\mathscr{T}_{\beta\alpha} = \langle\phi_\beta|V_\beta|\Psi_\alpha^{(+)}\rangle \tag{23}$$

the bracket notation is used to denote integration over all coordinates. β stands for any specific channel. Remember that ϕ_{beta} is the free solution and $\Psi_\alpha^{(+)}$ is the state vector of the whole system.

2.2 Asymptotic form of the complete wave function

Considering the elastic, inelastic and other break up channels, the complete wave function is written as:

$$\begin{aligned}\Psi_\alpha^{(+)} =&\Phi_\alpha\left[\exp\left(i\boldsymbol{k_\alpha}\cdot\boldsymbol{r}\right) + f_{\alpha\alpha}(\theta)\frac{e^{ik_\alpha r_\alpha}}{r_\alpha}\right] \\ &+ \sum_{\alpha'\neq\alpha}\Phi_\alpha f_{\alpha'\alpha}(\theta)\frac{e^{ik_{\alpha'} r_\alpha}}{r_\alpha} + \sum_\beta \Phi_\beta f_{\beta\alpha}(\theta)\frac{e^{ik_\beta r_\beta}}{r_\beta}\end{aligned} \tag{24}$$

3 Formalism

3.1 LS equation

The Schrodinger equation reads:

$$(E - H_\alpha - T_\alpha)|\Psi_\alpha\rangle = V_\alpha|\Psi_\alpha\rangle \tag{25}$$

including the free solution for boundary condition, we obtain the LS equation:

$$|\Psi_\alpha^{(+)}\rangle = |\phi_\alpha\rangle + \frac{1}{E - H_\alpha - T_\alpha + i\epsilon} V_\alpha |\Psi_\alpha^{(+)}\rangle \tag{26}$$

similarly, we define:

$$|\Psi_\alpha^{(-)}\rangle = |\phi_\alpha\rangle + \frac{1}{E - H_\alpha - T_\alpha - i\epsilon} V_\alpha |\Psi_\alpha^{(-)}\rangle \tag{27}$$

with the complete Hamiltonian:

$$|\Psi_\alpha^{(+)}\rangle = |\phi_\alpha\rangle + \frac{1}{E - H + i\epsilon} V_\alpha |\phi_\alpha\rangle \tag{28}$$

this can be obtained with the identity:

$$\frac{1}{A} = \frac{1}{B} + \frac{1}{B}(B - A)\frac{1}{A} \tag{29}$$

the Moller operator is defined as:

$$|\Psi_\alpha^{(+)}\rangle = \left(1 + \frac{1}{E - H + i\epsilon} V_\alpha\right)|\phi_\alpha\rangle = \Omega_\alpha^{(+)}|\phi_\alpha\rangle \tag{30}$$

3.2 T-matrix

The T-matrix is defined as:

$$\mathscr{T}_{\beta\alpha} = \langle\phi_\beta|V_\beta|\Psi_\alpha^{(+)}\rangle \tag{31}$$

express the state of the whole system with Green's operator:

$$\mathscr{T}_{\beta\alpha} = \langle\phi_\beta|V_\beta\left(1 + \frac{1}{E - H + i\epsilon} V_\alpha\right)|\phi_\alpha\rangle \tag{32}$$

here we are concerned about the expectation of V_β or V_α:

$$\begin{aligned}
\langle\phi_\beta|V_\beta - V_\alpha|\phi_\alpha\rangle &= \langle\phi_\beta|(H - H_\beta - T_\beta) - (H - H_\alpha - T_\alpha)|\phi_\alpha\rangle \\
&= \langle\phi_\beta|(H_\alpha + T_\alpha) - (H_\beta + T_\beta)|\phi_\alpha\rangle \\
&= 0
\end{aligned} \tag{33}$$

so we can replace V_β with V_α:

$$\begin{aligned} \mathscr{T}_{\beta\alpha} &= \langle \phi_\beta | V_\alpha + V_\beta \frac{1}{E - H + i\epsilon} V_\alpha | \phi_\alpha \rangle \\ &= \langle \phi_\beta | \left(1 + V_\beta \frac{1}{E - H + i\epsilon} \right) V_\alpha | \phi_\alpha \rangle \\ &= \langle \Psi_\beta^{(-)} | V_\alpha | \phi_\alpha \rangle \end{aligned} \tag{34}$$

so we have two ways of writing the transition amplitude:

$$\mathscr{T}_{\beta\alpha} = \langle \phi_\beta | V_\beta | \Psi_\alpha^{(+)} \rangle = \langle \Psi_\beta^{(+)} | V_\alpha | \phi_\alpha \rangle \tag{35}$$

these are sometimes referred as prior and post form of T-matrix.